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***Abstract*—We report on recent efforts towards the development of a high order, non-conforming, discontinuous Galerkin method for the solution of the system of frequency domain Maxwell's equations in heterogeneous propagation media. This method is an extension of the low order one which was proposed in [1].**

I. Introduction

In the recent years, there has been an increasing interest in discontinuous Galerkin (DG) methods for solving the time-domain Maxwell equations [2]- [3] due to their ability to easily deal with unstructured meshes and heterogeneous media. The development of DG methods for solving the frequency domain Maxwell equations has been less impressive and mostly done in the context of the Maxwell eigenvalue problem [4]. Besides, a hp-adaptive DG method has been proposed in the context of the low-frequency time-harmonic Maxwell equations [5]. In this paper, we report on recent efforts towards the development of a discontinuous Galerkin method for the solution of the system of frequency domain Maxwell's equations in heterogeneous propagation media. This DGFD (Discontinuous Galerkin Frequency Domain) method is formulated on simplicial meshes (triangle in 2D and tetrahedron in 3D). Within each mesh element, the approximation of the electromagnetic field relies on an arbitrarily high order nodal polynomial interpolation. Moreover, as a first step towards the development of a hp-adaptive method, the approximation order p is allowed to vary across mesh elements resulting in a non-conforming DGFD method. We present preliminary results for the simulation of two-dimensional propagation problems.

II. DGFD method

We consider solving the non-dimensioned time-harmonic Maxwell equations in the first order form:

$$\mathrm{i}\omega\varepsilon_r \boldsymbol{E} - \operatorname{curl} \boldsymbol{H} = 0 \quad , \quad \mathrm{i}\omega\mu_r \boldsymbol{H} + \operatorname{curl} \boldsymbol{E} = 0, \qquad (1)$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the unknown electric and magnetic fields. Parameters ε_r and μ_r are respectively the complex-valued relative dielectric permittivity and the relative magnetic permeability; we consider here the case of linear isotropic media. Eq. (1) is solved in a bounded domain Ω. On the boundary $\partial\Omega = \Gamma_a \cup \Gamma_m$, the following boundary conditions are imposed: a perfect electric conductor condition on Γ_m: $\boldsymbol{n} \times \boldsymbol{E} = 0$, and a Silver-Müller (first-order absorbing condition) condition on Γ_a: $\boldsymbol{n} \times \boldsymbol{E} + \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{H}) = \boldsymbol{n} \times \boldsymbol{E}^{\mathrm{inc}} + \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{H}^{\mathrm{inc}})$. Vectors $\boldsymbol{E}^{\mathrm{inc}}$ and $\boldsymbol{H}^{\mathrm{inc}}$ represent the components of an incident electromagnetic wave and $\boldsymbol{n}$ denotes the unitary outward normal. Eq. (1) can be further rewritten under the following form:

$$\mathrm{i}\omega G_0 \boldsymbol{W} + G_x \partial_x \boldsymbol{W} + G_y \partial_y \boldsymbol{W} + G_z \partial_z \boldsymbol{W} = 0. \qquad (2)$$

Let Ω_h denote a discretization of the domain Ω into a union of conforming tetrahedral elements $\overline{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K$. We look for the approximate solution $\boldsymbol{W}_h = (\boldsymbol{E}_h, \boldsymbol{H}_h)^t$ of (2) in $V_h \times V_h$ where the functional space V_h is defined by $V_h = \left\{ \boldsymbol{U} \in [L^2(\Omega)]^3 \ / \ \forall K \in \mathcal{T}_h, \ \ \boldsymbol{U}_{|K} \in \mathbb{P}_p(K) \right\}$ where $\mathbb{P}_p(K)$ denotes a space of vectors with polynomial components of degree at most p over the element K. The discontinuous Galerkin discretization of system (2) leads to find $\boldsymbol{W}_h$ in $V_h \times V_h$ such that:

$$\begin{aligned}
&\int_{\Omega_h} (\mathrm{i}\omega G_0 \boldsymbol{W}_h)^t \, \overline{\boldsymbol{V}} dv \\
&+ \sum_{K \in \mathcal{T}_h} \int_K \left(\sum_{l \in \{x,y,z\}} G_l \partial_l (\boldsymbol{W}_h) \right)^t \overline{\boldsymbol{V}} dv \\
&+ \sum_{F \in \Gamma^m \cup \Gamma^a} \int_F \left(\frac{1}{2} (M_{F,K} - I_{FK} G_{\boldsymbol{n}_F}) \boldsymbol{W}_h \right)^t \overline{\boldsymbol{V}} ds \\
&- \sum_{F \in \Gamma^0} \int_F (G_{\boldsymbol{n}_F} [\![\boldsymbol{W}_h]\!])^t \, \{ \overline{\boldsymbol{V}} \} ds \\
&+ \sum_{F \in \Gamma^0} \int_F (S_F [\![\boldsymbol{W}_h]\!])^t \, [\![\overline{\boldsymbol{V}}]\!] ds \\
&= \sum_{F \in \Gamma^a} \int_F \left(\frac{1}{2} (M_{F,K} - I_{FK} G_{\boldsymbol{n}_F}) \boldsymbol{W}^{\mathrm{inc}} \right)^t \overline{\boldsymbol{V}} ds,
\end{aligned} \qquad (3)$$

$\forall \boldsymbol{V} \in V_h \times V_h$, where Γ^0, Γ^a and Γ^m respectively denote the set of interior (triangular) faces, the set of faces on Γ_a

and the set of faces on Γ_m. The unitary normal associated to the oriented face F is $\boldsymbol{n}_F$ and I_{FK} stands for the incidence matrix between oriented faces and elements whose entries are given by 0 if the face F does not belong to element K, 1 if $F \in K$ and their orientations match, -1 if $F \in K$ and their orientations do not match. We also define respectively the jump and the average of a vector $\boldsymbol{V}$ of $V_h \times V_h$ on a face F shared by two elements K and $\tilde{K}$: $[\![\boldsymbol{V}]\!] = I_{FK}\boldsymbol{V}_{|K} + I_{F\tilde{K}}\boldsymbol{V}_{|\tilde{K}}$ and $\{\boldsymbol{V}\} = \frac{1}{2}\left(\boldsymbol{V}_{|K} + \boldsymbol{V}_{|\tilde{K}}\right)$. Finally, the matrix S_F, which is hermitian positive, allows to penalize the jump of a field or of some components of this field on the face F and the matrix $M_{F,K}$ is a numerical flux which can be either a centered flux or all the upwind flux (see [1] for more details).

III. Numerical results

Numerical results are presented here for the solution of the 2D TMz Maxwell equations. We first illustrate the convergence properties of the conforming (*i.e.* p is the same for all elements $K \in \mathcal{T}_h$) DGFD-$\mathbb{P}_p$ method by considering the propagation of a plane wave (F=300 MHz) in vacuum. The computational domain is the unit square [0,1]x[0,1] discretized by non-uniform triangular meshes. The numerical convergence of the method is visualized on Fig. 1 and convergence orders are summarized in Tab. I. One can note that an optimal convergence order is obtained in the case of an upwind numerical flux function.

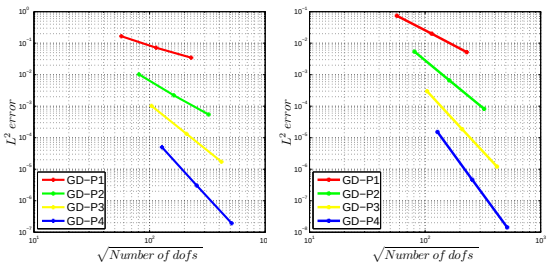

Fig. 1. Propagation of a plane wave in vacuum. Numerical convergence of the DGFD-$\mathbb{P}_p$ method: central flux (left) and upwind flux (right).

TABLE I
Propagation of a plane wave in vacuum. Convergence orders of the DGFD-$\mathbb{P}_p$ method.

Numerical flux	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$
Centered	1.1	2.1	2.9	4.0
Upwind	1.9	3.0	3.9	5.0

The second test problem that we consider is the scattering of a plane wave (F=300 MHz) by a dielectric cylinder. For that purpose, we make use of a highly non-uniform triangular mesh which consists of 2078 vertices and 3958 triangles. The relative permittivity of the inner cylinder is set to 2.25 while the vacuum is assumed for the rest of the domain. We compare the solutions obtained using a conforming DGFD-$\mathbb{P}_p$ method for $p = 1, 2, 3, 4$ and a non-conforming DGFD-$\mathbb{P}_{p_K}$ method and adopting a centered numerical flux function. In the latter case, the approximation order is defined empirically at the element level based on the triangle area resulting in a distribution for which the number of elements with $p = 1, 2, 3, 4$ is respectively equal to 1495, 2037, 243 and 183. For each method, the algebraic systems resulting from the discretization of the time-harmonic Maxwell equations is solved using an optimized sparse direct solver. In Tab. II we summarize the performances of the methods in terms of accuracy (L_2 error on the E_z component using the existing analytical solution for the considered problem), the CPU time and the memory overhead (for storing the L and U factors). These results clearly show the benefits resulting from a local definition of the approximation order, especially in terms of memory requirements and overall computational efficiency. The present work is currently proceeding towards the design of a p-adaptive solution strategy in the context of the proposed non-conforming DGFD-$\mathbb{P}_{p_K}$ method, and its extension to the 3D case.

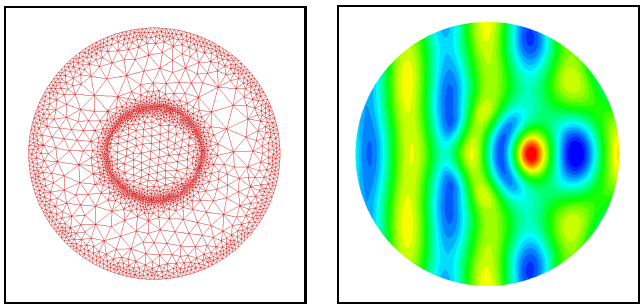

Fig. 2. Scattering of a plane wave by a dielectric cylinder. Non-uniform triangular mesh (top) and contour lines of E_z for the non-conforming DGFD-$\mathbb{P}_{p_K}$ method (bottom).

TABLE II
Scattering of a plane wave by a dielectric cylinder. Performance figures.

Method	L2 error on E_z	CPU	RAM (LU)
DGFD-$\mathbb{P}_1$	0.37977	1.3 sec	29 MB
DGFD-$\mathbb{P}_2$	0.58304	4.1 sec	84 MB
DGFD-$\mathbb{P}_3$	0.05527	7.9 sec	180 MB
DGFD-$\mathbb{P}_4$	0.05522	15.7 sec	317 MB
DGFD-$\mathbb{P}_{1,2,3,4}$	0.05586	3.7 sec	83 MB

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